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A New Method for Computing Toroidal Harmonics

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Abstract. A method for computing Legendre functions of integer order and half-odd degree is presented. The method is based on the theory of quadratic transformation of the argument, and is a generalization of Gauss' or Landen's transformation for computing elliptic integrals.

Legendre functions of integral order and half-odd degree are sometimes referred to as "Toroidal Harmonics." The latter name originates from the fact that these functions arise in the solution of Laplace's equation in toroidal coordinates ξ, η, ϕ. Separation of variables in this coordinate system leads, for the determination of one of the components, to the differential equation

$$(1) \qquad (1 - s^2)\frac{d^2y}{ds^2} - 2s\frac{dy}{ds} + \left[\left(\nu^2 - \frac{1}{4}\right) - \frac{\mu^2}{1 - s^2}\right]y = 0 \qquad (s = \cosh \xi),$$

while the remaining components have simple expressions of the form $e^{i\nu\eta}$, $e^{i\mu\phi}$, where μ and ν must be integers in order to obtain single-valued solutions. The solution of Eq. (1) can be given in terms of Legendre functions:

$$(2) \qquad y = AP^{\mu}_{\nu-1/2}(s) + BQ^{\mu}_{\nu-1/2}(s).$$

The functions $P^{\mu}_{\nu-1/2}(s)$ have the property that they are finite as $s \to 1$ but increase with increasing ν, while the functions $Q^{\mu}_{\nu-1/2}(s)$ are singular for $s \to 1$ but tend to zero as $\nu \to \infty$. Thus, in an exterior problem, it is customary to use the $Q^{\mu}_{\nu-1/2}$ with fixed μ and increasing ν.

For $\nu = 0$ and 1, $Q^{\mu}_{\nu-1/2}(s)$ can be expressed in terms of complete elliptic integrals, and hence higher degrees could, in theory, be computed by the known recurrence relation for these functions. However, as is well known, this recurrence relation is unstable, and poor or even meaningless values will be obtained for the higher degrees. This phenomenon also occurs in computing Bessel functions of higher order, and, as an alternative, J. C. P. Miller devised a scheme whereby the recurrence relation was employed in the backward direction (in which case no loss of accuracy is incurred), starting with values of the higher orders determined by some other means. Miller's algorithm was proposed by Rotenberg [1] as a means of computing toroidal harmonics, and used by Wright and Peterson [2] in preparing a brief table of these functions. The general theory behind this process has been the subject recently of some more detailed study by Gautschi [3] and Wimp [4]. A modification of the method used by the present author [5] employs a continued fraction to obtain the ratio of higher-degree terms.

The method described above has the disadvantage that all of the significant higher-degree functions must be computed first even if only a few of lower degree

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are needed. For s near 1, the convergence is very slow. As an example, when $s = 1.01$, over 100 terms of the continued fraction and functions up to degree 200 are required to be assured of 10-figure accuracy.

The present method is one which permits the functions to be computed directly starting with those of lowest degree and continuing as far as is desired. It is based on the theory of quadratic transformation, and is entirely analogous to the well-known Gauss or Landen transformation for elliptic integrals. Functions with arguments s are expressed in terms of those with argument s_1 where $s_1 > s$, and the transformation is repeated until an argument is obtained which is sufficiently large to allow the functions to be expressed by elementary functions. From these last results, those with the original argument can be calculated. The procedure can be applied with relatively no loss in accuracy.

The required transformation can be obtained from the two representations ([7, formula 3.671(2)], [8, Chapter IV, Section 5]).

(3) $$Q_{n-1/2}(s) = 2k^{n+1/2} \int_0^{\pi/2} \frac{\cos^{2n} \theta \, d\theta}{(1 - k^2 \cos^2 \theta)^{1/2}},$$

(4) $$Q_{n-1/2}(s_1) = k^{-1} \int_0^{\pi/2} \frac{\cos 2n\theta \, d\theta}{(1 - k^2 \cos^2 \theta)^{1/2}},$$

where

(5) $$s = (1 + k^2)/2k$$

and

$$s_1 = \frac{2}{k^2} - 1.$$

By using the relationship

(6) $$2^{2n-1} \cos^{2n} \theta = \left[\frac{1}{2}\binom{2n}{n} + \sum_{i=1}^{n} \binom{2n}{n-i} \cos 2i\theta\right],$$

we obtain

(7) $$Q_{n-1/2}(s) = \frac{[s + (s^2 - 1)^{1/2}]^{1/2-n}}{2^{2n-1}} \left[\binom{2n}{n} Q_{-1/2}(s_1) + 2 \sum_{i=1}^{n} \binom{2n}{n-i} Q_{i-1/2}(s_1)\right],$$

with

(8) $$s_1 = [s + (s^2 - 1)^{1/2}][s + 3(s^2 - 1)^{1/2}].$$

For $n = 0$ and 1 the above gives

(9)
$$Q_{-1/2}(s) = 2[s + (s^2 - 1)^{1/2}]^{1/2} Q_{-1/2}(s_1),$$
$$Q_{1/2}(s) = \frac{1}{[s + (s^2 - 1)^{1/2}]^{1/2}} [Q_{-1/2}(s_1) + Q_{1/2}(s_1)]$$

which, as can be easily verified, are equivalent to Gauss' transformation for elliptic integrals, upon making use of the relations

$$(10)\quad \begin{aligned} Q_{-1/2}(s) &= 2[s + (s^2 - 1)^{1/2}]^{-1/2} K[s - (s^2 - 1)^{1/2}], \\ Q_{1/2}(s) &= 2[s + (s^2 - 1)^{1/2}]^{1/2}\{ K[s - (s^2 - 1)^{1/2}] - E(s - (s^2 - 1)^{1/2})\}. \end{aligned}$$

It is clear that $s_1 > s$ and that, by repeating the transformation, s_m will ultimately be sufficiently large for the function $Q_{n-1/2}(s_m)$ to be approximated from the formula

$$(11)\qquad Q_{n-1/2}(s_m) \cong \frac{\sqrt{\pi}\,\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} [s_m + (s_m^2 - 1)^{1/2}]^{-(n+1/2)},$$

which is accurate to within a relative error of the order of $(1/2s_m)^2$.

Having found $Q_{n-1/2}(s_m)$, Eq. (7) may then be used to find $Q_{n-1/2}(s_{m-1})$, continuing until the required values of $Q_{n-1/2}(s)$ are obtained.

Finally functions of higher order can be obtained recursively from those of zero order by means of the formula

$$(12)\qquad (s^2 - 1)^{1/2} Q^{\mu+1}_{\nu+1/2}(s) = (\nu - \mu + \tfrac{1}{2}) s Q^{\mu}_{\nu+1/2}(s) - (\nu + \mu + \tfrac{1}{2}) Q^{\mu}_{\nu-1/2}(s),$$

for $\nu = 0, 1, 2, \cdots$ and from

$$(s^2 - 1)^{1/2} Q^{\mu+1}_{-1/2}(s) = (\tfrac{1}{2} - \mu) Q^{\mu}_{1/2}(s) - (\tfrac{1}{2} + \mu) s Q^{\mu}_{-1/2}(s),$$

for $\nu = -1$.

Numerical Example. To calculate $Q_{n-1/2}(s)$ for $s = 1.01$ and $n = 0, 1, 2, 3, 4, 5$. We start by computing s_i according to Eq. (8); until a value is obtained which is sufficiently large for Eq. (11) to apply:

i	s_i
0	1.01
1	1.65316 88539 02
2	1.66369 41381 96 $\times$ 10
3	2.20930 07386 65 $\times$ 10^3
4	3.90480 73030 93 $\times$ 10^7
5	1.21980 16059 43 $\times$ 10^{16}

The values of $Q_{n-1/2}(s_5)$ as given by Eq. (11) will now be accurate to better than 30 figures:

n	$Q_{n-1/2}(s_5)$
0	2.01136 21555 09 $\times$ 10^{-8}
1	4.12231 41240 83 $\times$ 10^{-25}
2	—
3	—
4	—
5	—

(Note that functions for $n \geqq 2$ are of no significance, since their magnitude relative to $Q_{-1/2}$ is negligible.)

Equation (7) is now used to compute $Q_{n-1/2}(s_i)$ for the remaining values of the s_i:

n	$Q_{n-1/2}(s_4)$	$Q_{n-1/2}(s_3)$
0	3.55496 28891 00 $\times 10^{-4}$	4.72614 96885 28 $\times 10^{-2}$
1	2.27601 68512 56 $\times 10^{-12}$	5.34801 56392 27 $\times 10^{-6}$
2	2.18578 34535 01 $\times 10^{-20}$	9.07755 98659 40 $\times 10^{-10}$
3	—	1.71199 72588 50 $\times 10^{-13}$
4	—	3.39020 77090 78 $\times 10^{-17}$
5	—	6.90532 31240 82 $\times 10^{-21}$

n	$Q_{n-1/2}(s_2)$	$Q_{n-1/2}(s_1)$
0	5.44995 36858 06 $\times 10^{-1}$	1.87832 66753 09
1	8.19787 45943 47 $\times 10^{-3}$	3.21017 38150 62 $\times 10^{-1}$
2	1.84955 91684 76 $\times 10^{-4}$	8.14856 90453 27 $\times 10^{-2}$
3	4.63643 83364 18 $\times 10^{-6}$	2.29243 39889 94 $\times 10^{-2}$
4	1.22035 66435 80 $\times 10^{-7}$	6.76530 09967 33 $\times 10^{-3}$
5	3.30386 17771 74 $\times 10^{-9}$	2.05248 65657 25 $\times 10^{-3}$

Finally we get:

n	$Q_{n-1/2}(s)$
0	4.03166 87796 89
1	2.04931 84514 62
2	1.41585 92547 73
3	1.05843 74848 35
4	8.21280 86319 17 $\times 10^{-1}$
5	6.51426 26170 33 $\times 10^{-1}$

As noted earlier, over 200 terms would have been required to obtain comparable accuracy by either of the two methods described earlier in the paper.

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1. A. Rotenberg, "Calculation of toroidal harmonics," *Math. Comp.*, v. 14, 1960, pp. 274–276. MR **22** #6066.

2. E. B. Wright & P. I. Peterson, *Brief Table of Ring Functions*, NRL Memorandum Report 1969, U. S. Naval Research Laboratory, March 1966.

3. W. Gautschi, "Computational aspects of three-term recurrence relations," *SIAM Rev.*, v. 9, 1967, pp. 24–82. MR **35** #3927.

4. J. Wimp, *Recent Developments in Recursive Computation*, ARL 69-0104, Aerospace Research Laboratories, July 1969.

5. Henry E. Fettis, *Calculation of Toroidal Harmonics without Recourse to Elliptic Integrals*, Blanch Anniversary Volume, Aerospace Research Lab., U. S. Air Force, Washington, D.C., 1967, pp. 21–34. MR **35** #2460.

6. Henry E. Fettis & J. C. Caslin, *Tables of Toroidal Harmonics*, ARL 69-0025, Aerospace Research Laboratories, February 1969. MR **39** #6481.

7. I. S. Gradshteyn & I. M. Ryzhik, *Table of Integrals, Series, and Products*, Fizmatgiz, Moscow, 1963; English transl., Academic Press, New York, 1965. MR **28** #5198; MR **33** #5952.

8. W. Magnus & F. Oberhettinger, *Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik*, 2nd ed., Springer, Berlin, 1948; English transl., Chelsea, New York, 1949. MR **10**, 38; MR **10**, 532.